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CONCERNING LOCALLY HOMOTOPY NEGLIGIBLE SETS AND CHARACTERIZATION OF ℓ_2-MANIFOLDS

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Concerning locally homotopy negligible sets and characterization of ℓ_2- manifolds *)

by

H. Toruńczyk **)

ABSTRACT

Let $A \subset X$ be a set such that for every open subset U of X the inclusion $U \setminus A \to U$ is a weak homotopy equivalence. The following two facts are shown:

(A) If X is an ANR($\mathcal{M}$)-space then so is $X \setminus A$;

(B) if A is closed in X, X is complete-metrizable and $X \setminus A$ is an ℓ_2-manifold, then so is X.

We apply (B) to prove that if X is a separable complete ANR($\mathcal{M}$) without isolated points then the space of paths in X forms an ℓ_2-manifold.

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CONTENTS

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Initially the paper was intended to present the proofs of the two following facts which had been announced, or employed, in [31] and [32]:

(A) If X is a complete separable ANR(M)-space and A is a countable union of Z-sets in X then $X\setminus A \in$ ANR(M), and

(B) If X as above, A is a Z-set in X and $X\setminus A$ is an ℓ_2-manifold then X is is an ℓ_2-manifold too.

By a Z-set in X we mean here any closed set $A \subset X$ with the property that every map f: $[0,1]^\infty \to X$ is a uniform limit of $X\setminus A$-valued maps.

Using results of infinite-dimensional topology, (A) has a very short proof: by [31], the space $X \times \ell_2$ is an ℓ_2-manifold which clearly contains $A \times \ell_2$ as a countable union of Z-sets; thus, by [2], $X \times \ell_2\setminus A \times \ell_2$ is homeomorphic to $X \times \ell_2$ and hence $(X\setminus A) \times \ell_2$ and $X\setminus A$ are ANR(M)'s. However, the assumption of (A) seems to be too restrictive: for instance, (A) does not include the fact that if A is any subset of the boundary of the square $[0,1]^2$ (A need not be of type F_σ), then $[0,1]^2\setminus A$ is an ANR(M). (See FOX [13]). Therefore we prove here in section 3 a more general result than (A), namely

(A') If $X \in$ ANR(M) and $A \subset X$ is locally homotopy negligible in X (i.e. for every open set $U \subset X$ the inclusion $U\setminus A \to U$ is a weak homotopy equivalence), then $X\setminus A \in$ ANR(M).

Since the properties of *non-closed* locally homotopy negligible sets have never been explicitly formulated, we devote a section of the paper to present the basic facts concerning such sets (see section 2). We note that most of these facts as well as of the methods used in their proofs are similar to these of ANDERSON [14], EELLS & KUIPER [11] and HENDERSON [19] (see also EILENBERG & WILDER [12], SMALE [29], HAVER [16]; however, several technical changes are to be made if one wants to dispense of the assumption that A is closed and X is an ANR(M). The material of section 2 allows to strengthen the results of [4], [22] and [23] on cell-like mappings of metric spaces (see Appendix); also, we hope that the study of non-closed locally homotopy negligible sets in concrete spaces can be used to prove that these spaces are ANR(M)'s or infinite-dimensional manifolds.

The result (B) stated before is proved in section 5.

In the paper we discuss also an elementary characterization of ANR(M)'s which is used in the proof of (A') (See section 1). Let us note that (A') can also be established by using a characterization of DOWKER & HANNER [10]; nevertheless the result of section 1 seems to be of independent interest (for instance, it unifies earlier results of WOJDYSLAWSKI [34], DUGUNDJI [9], HIMMELBERG [20] and others).

Notation.

By I we denote the interval [0,1], by N the set of integers, and continuous functions are called "maps". A homotopy $h: X \times I \to Y$ is often denoted by $(h_t): X \to Y$, where $h_t(x) = h(x,t)$. "$(h_t): (X,A) \to (Y,B)$" means that $A \subset X, B \subset Y$ and $h_t(A) \subset B$ for all $t \in I$. All spaces are assumed to be normal, and if X is a metrizable space then ρ usually denotes a metric which induces the topology of X. $p_X: X \times Y \to Y$ denotes the natural projection. By cov(X) we denote the family of all open coverings of X. If K is an (abstract) simplicial complex then $|K|$ denotes its standard geometric realization, endowed with the CW-topology. By i,k,n we denote elements of $N \cup \{0\} \cup \{\infty\}$ and "$i < n + 1$" means "$i \le n$ if $n < \infty$ and $i \ne \infty$ if $n = \infty$".

1. A CHARACTERIZATION OF ANR(M)'s

If A and V are families of subsets of X and $B \subset X$, then we say that A refines V (respectively B refines V) if each element of A is contained in an element of V (resp. if {B} refines V). By A_V we denote the family of all sets $A \in A$ which refine V.

Suppose that X and Z are spaces, A is a family of subsets of X and that to certain sets $A \in A$ we have assigned a map f_A from a non-empty set $dom(f_A) \subset Z$ into X. Given $U \in cov(X)$, we shall say that $(\{f_A\},Z)$ is a U-fine admissible approximation to A if there is a $V \in cov(X)$ such that the following conditions are satisfied.

(i) if $A \in A_V$ then f_A is defined, $A \cup im(f_A)$ refines U and $F_A = dom(f_A)$ is a homotopy trivial subset of Z;

(ii) if $A,B \in A_V$ and $A \subset B$ then f_B is an extension of f_A.

We shall sometimes say that $(\{f_A : A \in \mathcal{A}_{\mathcal{V}}\}, Z)$ forms the approximation. An approximation $(\{f_A\}, Z)$ will be said to be continuous if it is $\mathcal{U}$-fine for all $\mathcal{U} \in \mathrm{cov}(X)$. If $Z = X$ and each f_A is an inclusion then we say that the approximation is trivial; trivial approximations will be denoted by $\{F_A\}$, where $F_A = \mathrm{dom}(f_A) \subset X$. Finally, let us say that a family $\mathcal{U}$ has trivial intersections if $U_1 \cap U_2 \cap \ldots \cap U_n$ is homotopy trivial for all $U_1, U_2, \ldots, U_n \in \mathcal{U}$.

The aim of this section is to prove the following

THEOREM 1.1. *The following conditions are equivalent for a metric space* X:

(a) $X \in \mathrm{ANR}(\mathcal{M})$.

(b) *There exists a space* E *such that* $X \times E$ *has an open basis with trivial intersections.*

(c) *There exist continuous admissible approximations to the family of all finite subsets of* X.

(d) *There exist arbitrarily fine admissible approximations to the family of all finite subsets of* $X \times (0,1]$.

For simplicity the family of all finite subsets of X will be denoted by $\mathcal{F}(X)$ and the family of all subsets of X by $\mathcal{S}(X)$.

Remark. The implication (c) ⇒ (a) of theorem 1.1 generalizes earlier results of DUGUNDJI [8] and HIMMELBERG [20] stating that metric spaces which admit "nice" equiconnecting functions are $\mathrm{ANR}(\mathcal{M})$'s; see also MILNOR [24]. In fact if λ is an equiconnecting function on X then, letting $A_1 = A$ and inductively $A_{n+1} = \{\lambda(x,y,t) : x \in A,\ y \in A_n,\ t \in I\}$ and $F_A = \bigcup_n A_n$, we get a trivial approximation to $\mathcal{S}(X)$ which is continuous in the situations considered in [8] and [20]. (Note that F_A is contractible whenever it is defined).

Remark. Admissible approximations to $\mathcal{F} = \mathcal{F}(X)$ can be obtained as follows. Let $\mathcal{U} \in \mathrm{cov}(X)$, let K denote the simplicial complex of all $\{x_1, \ldots, x_n\} \in \mathcal{F}_{\mathcal{U}}$ and suppose there is given a map $f \colon |K| \to X$. Then, letting $Z = |K|$ and $f_\sigma = f|_{|\sigma|}$ for $\sigma \in K = \mathcal{F}_{\mathcal{U}}$, we get an approximation to $\mathcal{F}$ which is continuous if for each $x \in X$ and a neighbourhood U of x there is a neighbourhood $V \subset U$ of x such that $f(|\sigma|) \subset U$ for all $\sigma = \{x_1, \ldots, x_n\} \subset V$. In particular, the "convex structures" of [26] yield continuous approximations of this type and therefore theorem 1.1 generalizes the results stating that spaces which admit convex (or similar) structures are ANR(M)'s (see HIMMELBERG [20] and WOJDYSLAWSKI [34]).

In the proof of theorem 1.1 we need the following lemmas:

LEMMA 1.2. *Let* Y *be a metric space and* Y_0 *its dense subset. If there are arbitrarily fine admissible approximations to* $F(Y_0)$ *then there are also arbitrarily fine admissible approximations to* $S = S(Y)$.

PROOF. Fix $U \in \text{cov}(Y)$, let $U_1 \in \text{cov}(Y)$ be a star-refinement of U and let $(\{f_A: A \in A_V\}, Z)$ be a U_1-fine admissible approximation to $F = F(Y_0)$. We assume without loss of generality that V refines U_1. Let $W \in \text{cov}(Y)$ be a locally finite star-refinement of V and let $N \in \text{cov}(Y)$ be a refinement of W such that each element of N intersects only finitely many elements of W. For each $W \in W$ pick an $y_W \in Y_0 \cap W$ and, given $S \in S_N$, let $g_S = f_{\hat{S}}$, where $\hat{S} = \{y_W: W \in W \text{ and } W \cap S \neq \emptyset\}$. It is easy to see that $(\{g_S: S \in S_N\}, Z)$ is a U-fine approximation to the family S. □

LEMMA 1.3. *Let* (Y,ρ) *be a metric space and assume that there exist arbitrarily fine admissible approximations to* $S = S(Y)$. *Then, given* $\alpha: Y \to (0,\infty)$, *there are a simplicial complex* K *and maps* $f: Y \to |K|$ *and* $g: |K| \to Y$ *such that* $\rho(gf(y),y) < \alpha(y)$ *for all* $y \in Y$.

PROOF. Replacing, if necessary, ρ by $\tilde{\rho}(y_1,y_2) = \rho(y_1,y_2) + |\alpha(y_1) - \alpha(y_2)|$, we may assume that $|\alpha(y_1) - \alpha(y_2)| \leq \rho(y_1,y_2)$ for $y_1,y_2 \in Y$. Let $U = \{B(y,\alpha(y)/4): y \in Y\}$ be the cover of Y by open balls and let $(\{f_S: S \in S_V\}, Z)$ be a U-fine admissible approximation to S, with V being locally finite. Let K denote the nerve of V and for each $\sigma = \{V_1,\ldots,V_n\} \in K$ let

$$I(\sigma) = \{(f_{V_1\cap\ldots\cap V_n}(z),z): z \in \text{dom}(f_{V_1\cap\ldots\cap V_n})\} \subset Y \times Z$$

Clearly, I is an anti-monotone function from K to the non-empty homotopy trivial subsets of Y × Z (i.e. if $\sigma_1 \subset \sigma_2 \in K$ then $I(\sigma_1) \supset I(\sigma_2)$).

Now let K' denote the barycentric subdivision of K and let

$i\colon |K| \to |K'|$ be the subdivision map. For each $\sigma \in K$ we denote by $\hat{\sigma}$ its barycenter; $\hat{\sigma}$ is then vertex of K'.

SUBLEMMA. *There is a map* $\tilde{g}\colon |K'| \to Y \times Z$ *such that*

$$(*) \qquad \tilde{g}(|\{\hat{\sigma}_1,\hat{\sigma}_2,\ldots,\hat{\sigma}_n\}|) \subset I(\sigma_1) \qquad \textit{for all } \sigma_1 \subset \sigma_2 \ldots \sigma_n \in K.$$

PROOF. For each vertex $\hat{\sigma}$ of K' choose a point $\tilde{g}_0(\hat{\sigma}) \in I(\sigma)$. Let $(L,\tilde{g})$ be a maximal (under the natural ordering) pair such that L is a subcomplex of K' containing all vertices of K' and $\tilde{g}$ extends $\tilde{g}_0$ and satisfies $(*)$; we shall show that $L = K'$. Assume the contrary and let $s \in K'\backslash L$ be a simplex of minimal dimension. Then $\dim(s) \geq 1$ and *) $|\dot{s}| \subset |L|$, whence $\tilde{g} \mid |\dot{s}| : |\dot{s}| \to Y \times Z$ is well-defined. Representing s as $\{\sigma_1,\ldots,\sigma_n\}$, where $\sigma_1 \subset \sigma_2 \ldots \subset \sigma_n \in K$, we infer from $(*)$ and the anti-monotony of I that $g(|\dot{s}|) \subset I(\sigma_1)$. Since the set $I(\sigma_1)$ is homotopy trivial, we may extend $\tilde{g} \mid |\dot{s}|$ to a $g_s\colon |s| \to I(\sigma_1)$. Clearly $(L \cup \{s\},\tilde{g} \cup g_s)$ exceeds $(L,\tilde{g})$ which is impossible; thus $L = K'$ and g is as required. □

PROOF OF LEMMA 1.3 (continued). Let $g = p_Y \circ g \circ i$. Given $y \in Y$ let $\{V_1,\ldots,V_n\} = \{V \in \mathcal{V}\colon y \in V\}$. Observe that, by $(*)$, we have

$$g(|\{V_1,\ldots,V_n\}|) \subset p_Y(\bigcup_{i\leq n} \operatorname{im}(f_{V_i})).$$

Since $\operatorname{im}(f_{V_i}) \cup Y_i$ refines $\mathcal{U}$ for $i=1,2,\ldots,n$, we infer that $g(|\{V_1,\ldots,V_n\}|)$ is contained in the star of y in $\mathcal{U}$. Therefore, if $f\colon Y \to |K|$ is induced by a partition of unity $\{\lambda_V\colon V \in \mathcal{V}\}$ with each λ_V vanishing outside V, then $gf(y) \in st(y,\mathcal{U})$ for all $y \in Y$. This easily yields $\rho(gf(y),y) < \alpha(y)$ for all $y \in Y$. □

The following lemma is actually a special case of a theorem of DOWKER & HANNER [10, p. 105], we include however a short proof of it that will be

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*) $\dot{s}$ denotes the boundary of s.

used later.

LEMMA 1.4. *Let* (X,ρ) *be a metric space and assume that there are a simplicial complex* K *and maps* $f\colon X \times (0,1] \to |K|$ *and* $g\colon |K| \to X$ *such that* $\rho(gf(x,t),x) < t$ *for all* $(x,t) \in X \times (0,1]$. *Then* $X \in ANR(M)$.

PROOF. Let $h\colon A \to X$ be a map of a closed set A of a metric space (B,ρ_B); we shall construct a neighbourhood extension of h.

To this purpose let $u = h \times id\colon A \times (0,1] \to X \times (0,1]$. Since simplicial complexes are neighbourhood extensors for metric spaces, $f \cdot u$ admits an extension $v\colon U \to |K|$, where $U \subset B \times (0,1]$ is an open set containing $A \times (0,1]$ (see [21], p. 105). Without loss of generality we may assume that U is contained in the set $\{(b,t) \in U$: there is an $a \in A$ with $\rho_B(a,b) < t$ and $\rho(gv(b,t),h(a)) < t\}$, which, by our assumptions, is a neighbourhood of $A \times (0,1]$. Let $\lambda\colon B \to [0,1]$ be such that $\lambda|A = 0$ and $\{(b,\lambda(b)):b \in B\setminus A\} \subset U \cup (B\setminus A) \times \{1\}$. ($\lambda$ can easily be constructed by using TIETZE's theorem and the fact that for each $a \in (0,1]$ there is a closed neighbourhood W of A in B with $W \times [a,1] \subset U$). We let $V = \{b \in B: \lambda(b) < 1\}$ and define $\bar{h}\colon V \to X$ by

$$\bar{h}(b) = \begin{cases} h(b) & \text{if } b \in A \\ gv(b,\lambda(b)) & \text{if } b \in V\setminus A. \end{cases}$$

It is easily seen that $\bar{h}$ is continuous. □

Now we can complete the proof of theorem 1.1. To show that (a) ⇒ (c) consider X as a closed subset of a convex set Z in a normed linear space [21,p.81] and for sufficiently small sets $A \subset X$ let $F_A = \text{conv}(A) \subset Z$ and $f_A = r|F_A$, where r is a neighbourhood retraction onto X. The implication (c) ⇒ (d) is a consequence of the fact that any pair of continuous admissible approximations to $F(X)$ and to $F(Y)$ induces a continuous product approximation to $F(X{\times}Y)$. Further, (d) ⇒ (a) by lemmas 1.2-1.4, and (a) ⇒ (b) by a result of [31] stating that if $X \in ANR(M)$ then there exists a normed linear space E with $X \times E$ homeomorphic to an open subset of E. Finally, (b) ⇒ (a) follows from the implication (c) ⇒ (a) and the following fact applied to $Y = X \times E$.

SUBLEMMA. *If* Y *has a base* $\mathcal{U}$ *with trivial intersections then there exist trivial continuous approximations to* $\mathcal{S}(Y)$.

PROOF. For each $n \in N$ let $\mathcal{V}_n \in \mathrm{cov}(X)$ be a locally finite refinement of $\mathcal{U}$ with $\mathrm{diam}_\rho V < 1/n$ for all $V \in \mathcal{V}_n$. Let $\mathcal{V} = \cup_n \mathcal{V}_n$, let $V \to U_V$ be a function of $\mathcal{V}$ into $\mathcal{U}$ such that $V \subset U_V$ for all $V \in \mathcal{V}$, and for sufficiently small $A \subset Y$ let $F_A = \cap_{V \in \mathcal{V}(A)} U_V$, where $\mathcal{V}(A) = \{V \in \mathcal{V}: A \in V\}$. It is easy to see that $\mathcal{V}(A)$ is finite for all $A \subset Y$ and F_A is a continuous trivial approximation to $\mathcal{S}(Y)$. □

COROLLARY 1.5. *Let* X *be a separable complete metric space which is* ℓ_2*-stable (i.e.* $X \times \ell_2 \cong X$*). Suppose further that there exist arbitrarily fine admissible approximations to* $\mathcal{F}(X_0)$*, where* $X_0 \subset X$ *is a dense set. Then* X *is an* ℓ_2*-manifold.*

PROOF. By a theorem of KLEE we have $\ell_2 \times (0,1] \cong \ell_2$ (see [31]) and therefore $X \times \ell_2 \times (0,1] \cong X \times \ell_2$ and $X \times (0,1] \cong X$. Hence, by theorem 1.1 and lemma 1.2, $X \in \mathrm{ANR}(\mathcal{M})$ and thus, by [31], $X \times \ell_2$ is an ℓ_2-manifold. Since $X \times \ell_2 \cong X$, the result follows. □

Clearly, the conditions of corollary 1.5 are also necessary for a connected space X to be an ℓ_2-manifold (recall that each separable ℓ_2-manifold is ℓ_2-stable and is homeomorphic to an open subset of ℓ_2, see [3] and [31]).

2. LOCALLY HOMOTOPY NEGLIGIBLE SETS

DEFINITION 2.1. A set $A \subset X$ will be said to be *locally* n*-negligible* if, given $x \in X$, $k < n + 1$ and a neighbourhood U of x, there is a neighbourhood $V \subset U$ of x such that for each $f: (I^k, \partial I^k) \to (V, V \setminus A)$ there is a homotopy $(h_t): (I^k, \partial I^k) \to (U, U \setminus A)$ with $h_0 = f$ and $h_1(I^k) \subset U \setminus A$. Locally ∞-negligible sets will also be called *locally homotopy negligible* (briefly: l.h. negligible).

The aim of this section is to discuss certain properties of l.h. negligible sets; we formulate the corresponding results for locally n-negligible sets with $n < \infty$ only if their proofs require no extra work.

REMARK 2.2. Let A be a locally n-negligible set in X. Then

(a) For every space E, $A \times E$ is locally n-negligible in $X \times E$.

(b) For every open set $U \subset X$, $U \cap A$ is locally n-negligible in U.

THEOREM 2.3. *Let* $A \subset X$, *where* X *is normal. The following conditions are equivalent*:

(a) A *is locally n-negligible in* X

(b) *Given* $\varepsilon > 0$, *a pseudometric* ρ *on* X *and a map* $f:(|K|,|L|) \to (X,X\backslash A)$, *where* (K,L) *is a finite simplicial pair with* $\dim(K) \leq n$, *there is a homotopy* (h_t): $|K| \to X$ *such that* $h_0 = f$, $h_1(|K|) \subset X\backslash A$, $h_t(x) = f(x)$ *for* $(x,t) \in |L| \times I$, *and* $\rho(h_t(x),f(x) < \varepsilon$ *for* $(x,t) \in |K| \times I$.

(c) *Given a simplicial pair* (K,L) *with* $\dim(K) < n + 1$, *a pseudometric* ρ *on* X *and maps* α: $|K| \to (0,\infty)$ *and* f: $|K| \times \{0\} \cup |L| \times I \to X$ *with* $\rho(f(x,t),f(x,0)) < \alpha(x)$ *and* $f(x,1) \notin A$ *for all* $(x,t) \in |L| \times I$, *there is an* f: $|K| \times I \to X$ *which extends* f *and satisfies* $\rho(f(x,t),f(x,0)) < \alpha(x)$ *and* $f(x,1) \notin A$ *for all* $(x,t) \in |K| \times I$.

(d) *For each open* $U \subset X$ *and* $i < n + 1$ *the relative homotopy group* $\pi_i(U,U\backslash A)$ *vanishes*.

(e) *Each* $x \in X$ *has a basis* $\mathcal{U}_x$ *of open neighbourhoods with* $\pi_i(U,U\backslash A) = 0$ *for all* $U \in \mathcal{U}_x$ *and* $i < n + 1$.

PROOF. (a) ⇒ (b). Let (b_p) denote the condition obtained from (b) with "$\dim(K) < n + 1$" replaced by "$\dim(K) \leq p$"; we shall show that (a) ⇒ (b_p) for $0 \leq p < n + 1$. Assume (a) ⇒ (b_{p-1}) has been established (evidently (a) ⇒ (b_0)) and let K,L,f and ρ be as in (b_p). Given $\varepsilon \in (0,\frac{1}{2})$ cover the compact set $f(|K|)$ by open sets $V_1,\ldots,V_k$ such that for each g: $(I^p,\partial I^p) \to (V_i,V_i\backslash A)$, where $1 \leq i \leq k$, there is a homotopy $h = (h_t)$: $(I^p,\partial I^p) \to (X,X\backslash A)$ with $h_0 = g$, $h_1(I^p) \cap A = \emptyset$ and $\operatorname{diam}_\rho \operatorname{im}(h) < \varepsilon$. Let a subdivision (K',L') of (K,L) be so fine that $\{f(|\sigma|): \sigma \in K'\}$ refines $\{V_1,\ldots,V_k\}$ (we identify $|K'|$ with $|K|$), and for each $\sigma \in K'$ let $\lambda_\sigma : X \to [0,1]$ be a map that is 1 on $f(|\sigma|)$ and 0 outside a $V^\sigma \in \{V_1,\ldots,V_k\}$.

Let $d(x,y) = \sum_{\sigma\in K'}|\lambda_\sigma(x) - \lambda_\sigma(y)| + \rho(x,y)$, for $x,y \in X$, and let M be the union of L' and p − 1 skeleton of K'. By (b_{p-1}), there is a homotopy $(\tilde{f}_t)$: $|M| \to X$ such that $\tilde{f}_0 = f \mid |M|$, $\tilde{f}_1(|M|) \cap A = \emptyset$, $\tilde{f}_t \mid |L| = f||L|$ and $d(\tilde{f}_t(x),f(x)) < \varepsilon$ for all $(x,t) \in |M| \times I$. For each $\sigma \in K'\backslash M$ denote $T_\sigma = |\dot{\sigma}| \times I \cup |\sigma| \times \{0\}$ and let $\tilde{f}^\sigma$ be the map induced by $\tilde{f}$ on $|\dot{\sigma}| \times [0,1]$ and by f on $|\sigma| \times \{0\}$. Then $(T_\sigma,|\dot{\sigma}|\times\{1\}) \cong (I^p,\partial I^p)$ and $\tilde{f}^\sigma(T_\sigma) \subset V_\sigma$ for all $\sigma \in K'\backslash M$ and therefore, by our construction, there are homotopies (g^σ_t): $(T_\sigma,|\dot{\sigma}|\times\{1\}) \to (X,X\backslash A)$ such that $g^\sigma_0 = \tilde{f}^\sigma$, $g^\sigma_1(T_\sigma) \cap A = \emptyset$ and $\mathrm{diam}_\rho \mathrm{im}(g^\sigma) < \varepsilon$ for each $\sigma \in K'\backslash M$. Then the g^σ's induce maps h^σ: $|\sigma| \times I \to X$ such that $h^\sigma|T_\sigma = \tilde{f}^\sigma$, $h^\sigma(|\sigma|\times\{1\}) \subset X\backslash A$ and $\mathrm{diam}_\rho \mathrm{im}(h^\sigma) < \varepsilon$ (we take $h^\sigma = f^\sigma u^\sigma$, where u^σ is a homeomorphism of $|\sigma| \times I$ onto $T_\sigma \times I$ such that $u^\sigma(x) = (x,0)$ for $x \in T_\sigma \subset |\sigma| \times I$ and $u^\sigma(|\sigma|\times\{1\}) = T_\sigma \times \{1\} \cup |\sigma| \times \{1\}\times I$). We let

$$h_t(x) = \begin{cases} h^\sigma(x,t) & \text{if } x \in |\sigma| \text{ and } \sigma \in K'\backslash M \\ \tilde{f}_t(x) & \text{if } x \in |M|. \end{cases}$$

(b) ⇒ (c). By the KURATOWSKI-ZORN lemma it suffices to consider the case where $K = \sigma$ is a simplex and $L = \dot{\sigma}$. Assume that $|\sigma|$ is embedded in a euclidean space and for each $A \subset |\sigma|$ denote by λA the image of A under the λ-homothethy with respect to the barycenter 0 of $|\sigma|$. Let $\varepsilon > 0$ satisfy $\varepsilon < \min\{\alpha(x) : x \in |\sigma|\}$ and $\varepsilon < \min\{\alpha(x) - \rho(f(x,t),f(x,0)) : (x,t) \in |\dot{\sigma}| \times I\}$. Set $T = |\sigma| \times \{0\} \cup |\dot{\sigma}| \times I$; by (b) there is an ε-homotopy $w: T \times I \to X$ such that $w_1(T) \cap A = \emptyset$, $w_0 = f$ and $w_t(x) = f(x,1)$ for $x \in |\sigma| \times \{1\}$. Now, for each $x \in |\sigma|\backslash\{0\}$ let $A(x) = \{(\lambda x,0) : \lambda \in [1,\mu]\} \cup \{(\mu x,t) : t \in I\}$, where $\mu \geq 1$ is chosen so that $\mu x \in |\dot{\sigma}|$. Then the inequality

$$\sup\{\rho(f(y),f(x,0)) : y \in A(x)\} < \alpha(x) - \varepsilon \tag{2.1}$$

holds for all $x \in |\dot{\sigma}|$ and therefore, by compactness, there is a $\lambda \in (0,1)$ such that (2.1) holds for all $x \in |\sigma|\backslash\lambda|\sigma|$. Let (u_t): $|\sigma| \to T \times I$ be a homotopy such that: (i) $u_t(x) = ((x,t),0)$ if $(x,t) \in |\dot{\sigma}| \times I \cup |\sigma| \times \{0\}$;
(ii) $u_t(x) = ((x,0),t)$ if $(x,t) \in \lambda|\sigma| \times I$;
(iii) $u_1(|\sigma|) \subset T \times \{1\} \cup |\dot{\sigma}| \times \{1\} \times I$; and (iv) $p_T u_t(x) \in A(x)$ if

$(x,t) \in (|\sigma|\setminus\lambda|\sigma|) \times I$. Then $f\colon |\sigma| \times I \to X$ defined by $f(x,t) = w(u_t(x))$ is the required extension of f.

The implications (c) ⇒ (b) and (d) ⇒ (e) ⇒ (a) are evident. To prove that (b) ⇒ (d), fix $f\colon (I^k,\partial I^k) \to (U,U\setminus A)$, where $U \subset X$ is open and $k < n + 1$. Let $\lambda\colon X \to I$ be a function that is 0 on $X\setminus U$ and 1 on $f(I^k)$ and let $(h_t)\colon I^k \to X$ be a homotopy such that $h_0 = f$, $h_1(I^k) \cap A = \emptyset$, $h_t(x) = f(x)$ for $x \in \partial I^k$ and $|\lambda f_t(x) - \lambda f(x)| < 1/2$ for $x \in I^k$ (all $t\in I$). Then $h_t(I^k) \subset U$ for all $t \in I$ and hence f is trivial in $\pi_k(U,U\setminus A)$. □

If (X,ρ) is a metric space and $h\colon M \times I \to X$ and $\alpha\colon M \times I \to [0,\infty)$ are maps then we shall say that h is an α-homotopy if $\rho(h_t(x),h_0(x)) \leq \alpha(x,t)$ for all $(x,t) \in M \times I$.

<u>THEOREM 2.4</u>. *Let* A *be an l.h. negligible set in a metric space* (X,ρ) *and let* $f\colon M \to X$ *be a map of an* ANR(*M*)*-space* M. *Then, given* $\alpha\colon M \times [0,1] \to [0,\infty)$ *with* $\alpha(x,t) > 0$ for $(x,t) \in f^{-1}(A) \times (0,1]$, *there is an* α*-homotopy* $(h_t)\colon M \to X$ *such that* $h_0 = f$ *and* $h_t(M) \subset X\setminus A$ *for* $t \in (0,1]$.

We first consider a special case of theorem 2.4.

<u>SUBLEMMA</u>. *Let* X, M, A *and* f *be as above and let* $\gamma\colon M \to (0,\infty)$. *Then there exists a* $g\colon M \to X\setminus A$ *such that* $\rho(g(x),f(x)) < 4\gamma(x)$ *for* $x \in M$.

<u>PROOF</u>. Let $\mathcal{U} \in \mathrm{cov}(M)$ be so fine that $\mathrm{diam}_\rho f(U) < \sup\{\gamma(x) : x \in U\} < 2\inf\{\gamma(x) : x \in U\}$ for all $U \in \mathcal{U}$, and let a simplicial complex K and maps $u_1\colon M \to |K|$, $u_2\colon |K| \to M$ be such that for each $x \in M$ there is a $U \in \mathcal{U}$ with $\{u_2 u_1(x),x\} \subset U$ (cf.[11,p.138]). By 2.3 there exists a $g_0\colon |K| \to X\setminus A$ such that $\rho(g_0(y),fu_2(y)) < \gamma u_2(y)$ for all $y \in |K|$. We let $g = g_0 u_1$. □

<u>PROOF OF THEOREM 2.4</u>. Let $X' = X \times (0,1]$, $A' = A \times (0,1]$, $M' = \alpha^{-1}(0,\infty)$ and let $f'\colon M' \to X'$ be defined by $f'(x,t) = (f(x),t)$. By 2.2 and the sublemma, there exists a $g\colon M' \to X'\setminus A'$ such that $\rho'(g(x,t),f'(x,t)) < \min(t,\alpha(x,t))$ for $(x,t) \in M'$, where $\rho'((x,t),(y,s)) = \rho(x,y) + |t - s|$. We let $h_t(x) = p_X g(x,t)$ if $(x,t) \in M'$ and $h_t(x) = f(x)$ if $(x,t) \in M \times \{0\} \cup \alpha^{-1}(0)$. □

REMARK 2.5. Assume that X,M and f are as in 2.4 and that A is locally n-negligible in X. If $\dim(M) \leq n - 1$, then the assertion of theorem 2.4 still holds. If $\dim(M) \leq n$, then for every $\beta: X \to [0,\infty)$ with $\beta|A > 0$ there is a homotopy $(h_t): M \to X$ such that $h_0 = f$, $h_1(M) \subset X\setminus A$ and $\rho(h_t(x),f(x)) < \beta(f(x))$ for all $(x,t) \in M \times I$. (We apply the proof of theorem 2.4 and the fact that if $M_1 \in ANR(M)$ is of covering dimension n then there are an n-dimensional simplicial complex K and maps $M_1 \overset{u_1}{\to} |K| \overset{u_2}{\to} M_1$ such that u_2u_1 is homotopic to id by means of a small homotopy).

COROLLARY 2.6. *If* A *is an l.h. negligible set in a metric space* X *and* A' *is a subset of* A, *then* A' *is also l.h. negligible in* X.

PROOF. Let an open set $U \subset X$ and $f: (I^n,\partial I^n) \to (U,U\setminus A')$ be given, and let $\varepsilon = \rho(f(I^n),X\setminus U)$. By theorem 2.4, there exists an ε-homotopy $(h_t): I^n \to X$ such that $h_0 = f$ and $h_t(I^n) \cap A = \emptyset$ for $t \in (0,1]$. Then $(h_t): (I^n,\partial I^n) \to (U,U\setminus A')$ satisfies the condition in definition 2.1. □

COROLLARY 2.7. *Let* $A_1,A_2,\ldots$ *be closed l.h. negligible sets in* X. *If* X *is complete-metrizable then* $A = \bigcup_i A_i$ *is l.h. negligible in* X.

PROOF. Fix $n \geq 0$ and consider the space Y of all maps of $I^n \times (0,1]$ into X, equipped with the "fine topology" generated by all sets $V(g,\alpha) = \{h \in Y : \rho(h(x),g(x) < \alpha(x)\}$, where ρ is a fixed complete metric on X, $g \in Y$ and α is a map from $I^n \times (0,1]$ into $(0,\infty)$.

By theorem 2.4, all the sets $Y_n = \{g \in Y : im(g) \cap A_n = \emptyset\}$ are dense and open in Y. Moreover, it is easy to verify that Y has the Baire property (cf.[30]) and therefore $Y_\infty = \bigcup_n Y_n$ is dense in Y. Thus for each $f: I^n \to X$ there is an $h \in Y_\infty$ with $\rho(h(x,t),f(x)) < \varepsilon t$ for all $(x,t) \in I^n \times (0,1]$; this easily completes the proof. □

We conclude this section by giving a condition for a set $A \subset X$ to be locally n-negligible. Following [12] we say that $B \subset X$ is LC^n rel. X at a point $x \in X$ if, given $k < n + 2$ and a neighbourhood U of x, there is a neighbourhood U of x, there is a neighbourhood $V \subset U$ of x such that each $f: \partial I^k \to B \cap V$ extends to an $\bar{f}: I^k \to B \cap U$.

THEOREM 2.8. (Compare [12]). *Let* X *be a metric space and let* $A \subset X$ *be a set such that* $X \setminus A$ *is dense in* X *and is* LC^n *rel.* X *at each point of* $\bar{A}$. *If* $n < \infty$ *then* A *is locally n-negligible in* X *and each map* $f: I^{n+1} \to X$ *can be approximated by maps* $f': I^{n+1} \to X \setminus A$ *which coincide with* f *on an arbitrary given compact subset of* $f^{-1}(X \setminus \bar{A})$.

PROOF. Let $f: K \to X$ be a fixed map of a compact polyhedron K. Denoting $L = f(K) \cap \bar{A}$ we let for any map $g: Z \to X$ of a compact space Z

$$\delta(g) = \mathrm{diam}_\rho g(Z) + \sup\{\rho(g(z), L) : z \in Z\},$$

and we say that g is λ-small if $\delta(g) < \lambda$. By a standard compactness argument there exist a $\lambda_0 > 0$ and a function $\varepsilon: (0,\lambda_0] \to (0,\infty)$ with $\lim_{\lambda \to 0} \varepsilon(\lambda) = 0$ and such that each λ-small $g: \partial I^k \to X \setminus A$ admits an $\varepsilon(\lambda)$-small extension $\bar{g}: I^k \to X \setminus A$ $(k = 0,1,\ldots,n+1)$. Without loss of generality we can assume that $\lambda_0 > 3$ and that ε is non-decreasing.

Claim A. *If* $\dim(K) \leq n+1$ *then, for every* $\mu \in (0,1]$, *there exists a* $g: K \to X \setminus A$ *such that* $\hat{\rho}(f,g) < \varepsilon(3\mu) + 3\mu$ *and* $g(x) = f(x)$ if $\rho(f(x),L) > \mu$. *)

PROOF. We use induction on $\dim(K)$. Suppose claim A holds if $\dim(K) \leq p$ (it does if $\dim(K) = 0$) and assume $\dim(K) = p+1 \leq n+1$. Let T be a triangulation of K such that $\mathrm{diam}_\rho f(|\sigma|) < \mu$ for any simplex $\sigma \in T$ and let S denote the p-skeleton of T. Let $g_0: |S| \to X \setminus A$ be a map such that $\hat{\rho}(g_0, f \mid |S|) < \mu$ and such that $g_0(x) = f(x)$ if x lies in a simplex of T which is disjoint from $f^{-1}(\bar{A})$. Now, let $\sigma \in T$ be any $(p+1)$-simplex. If $|\sigma| \cap f^{-1}(A) \neq \emptyset$ then $g_0 \mid |\dot{\sigma}|$ is 3μ-small and therefore it admits an $\varepsilon(3\mu)$-small extension $g^\sigma: |\sigma| \to X \setminus A$. If $|\sigma| \cap f^{-1}(\bar{A}) = \emptyset$ then put $g^\sigma = f \mid |\sigma|$. Clearly, g_0 and the g^σ's induce a required $g: K \to X \setminus A$. □

Claim B. *If* $\dim(K) \leq n$ *then there is a homotopy* $(h_t): K \to X$ *with* $h_0 = f$ *and* $h_t(K) \subset X \setminus A$ *for* $t \in (0,1]$.

PROOF. Define $\varepsilon_0(\mu) = \varepsilon(3\mu)$ and inductively $\varepsilon_{j+1}(\mu) = 2\varepsilon_j(\mu) + 3\mu$; we may

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*) By $\hat{\rho}$ we denote the sup-metric induced by ρ.

assume that $\sup\{\mu : \text{all } \varepsilon(\mu), \varepsilon_1(\mu), \ldots, \varepsilon_n(\mu) \text{ are defined}\} > 1$. Let T_i, $i \geq 1$, be triangulations of K such that T_{i+1} is a subdivision of T_i and $\operatorname{diam}_\rho f(|\sigma|) < 2^{-i}$ for all $\sigma \in T_i$ and $i \geq 1$, and for each $i \geq 1$ let $g_i \colon K \to X \setminus A$ be a map such that $\hat{\rho}(g_i, f) < 2^{-i}$ and $g_i(x) = f(x)$ if x lies in a simplex of T_i disjoint from $f^{-1}(\bar{A})$. We shall find our desired homotopy in such a way that $h_t = g_i$ if $t = 2^{-i+1}$, $i=1,2,\ldots$ To make this possible it suffices to construct for each $i \geq 1$ a $h^i \colon K \times I \to X \setminus A$ with $h^i_0 = g_i$, $h^i_1 = g_{i+1}$ and $\hat{\rho}(h^i_t, f) < \varepsilon_n(2-^i) + 2^{-i}$ for $t \in I$.

To this aim, fix i and let v be a vertex of T_i. If $v \in f^{-1}(\bar{A})$ then $g_i(v)$ and $g_{i+1}(v)$ can be joined by an $\varepsilon(3 \cdot 2^{-i})$-small path lying in $X \setminus A$, and if $v \notin f^{-1}(\bar{A})$ then this path can be taken as the constant one. Proceeding in this way with all vertices v of T_i we get a $h^{i,0} \colon |T^0_i| \times I \cup K \times \{0,1\} \to X$ with $h^{i,0}|K \times \{0\} = g_i$ and $h^{i,0}|K \times \{1\} = g_{i+1}$ (by T^k_i we denote the k-skeleton of T_i). Now let $\sigma \in T^1_i$. If $|\sigma| \cap f^{-1}(A) \neq \emptyset$ then $h^{i,0} \mid |\dot{\sigma}| \times I \cup |\sigma| \times \{0,1\}$ is an $2\varepsilon_0(2^{-i}) + 3 \cdot 2^{-i}$-small map of a 1-sphere and therefore it can be extended to an $\varepsilon_1(2^{-i})$-small map of $|\sigma| \times I$ into $X \setminus A$; if $|\sigma| \cap f^{-1}(\bar{A}) = \emptyset$ then this extension can be taken to be constant on all intervals $\{x\} \times I$, $x \in |\sigma|$. In this way one gets an $h^{i,1} \colon |T^1_i| \times I \cup K \times \{0,1\} \to X$ which extends $h^{i,0}$ and has the property that $h^{i,1} \mid |\sigma| \times I$ is $\varepsilon_1(2^{-i})$-small for all $\sigma \in T^1_i$. Inductively, we get maps $h^{i,j} \colon |T^j_i| \times I \cup K \times \{0,1\} \to X$, $j=1,2,\ldots,n$, such that $h^{i,j+1}$ extends $h^{i,j}$ and $h^{i,j} \mid |\sigma| \times I$ is $\varepsilon_j(2^{-i})$-small for all $\sigma \in T^j_i$. We let $h^i = h^{i,n}$. □

Clearly the claims A and B imply the assertion of theorem 2.8. □

REMARK 2.9. If $Y \subset X$ is a dense set which is uniformly LC^∞ in a metric of X, then Y is LC^∞ rel. X at each $x \in X$ and, hence, $X \setminus Y$ is l.h. negligible in X. (A version of this remark was made by EILENBERG & WILDER [12] and various forms of it were applied by HAVER [16],[17] in the study of function spaces.)

3. LOCALLY HOMOTOPY NEGLIGIBLE SETS IN ANR(M)'s AND LC^∞-SPACES

THEOREM 3.1. *Let* $X \in$ ANR(M) *and let* A *be a locally homotopy negligible* *set in* X. *Then* $X\backslash A \in ANR(M)$.

PROOF. By theorem 1.1, there exists a space E such that $X \times E$ has an open basis (say $\mathcal{U}$) with trivial intersections. Then $A \times E$ is l.h. negligible in $X \times E$ and therefore the basis $\{U\backslash A \times E : U \in \mathcal{U}\}$ of $(X\backslash A) \times E$ has trivial intersections. Hence $(X\backslash A) \times E$ and $X\backslash A$ are ANR(M)'s (we use theorem 1.1 again). □

PROPOSITION 3.2. *Let* $X \in ANR(M)$ *and let* A *be a locally n-negligible set in* X. *If* $\dim(X) \le n$ *then* A *is locally* ∞*-negligible in* X.

PROOF. Fix $f: (I^k, \partial I^k) \to (X, X\backslash A)$ and $\varepsilon > 0$. By remark 2.5 there exists an ε-homotopy $(h_t): X \to X$ such that $h_1(X) \subset X\backslash A$ and $h_t(x) = x$ if $(x,t) \in X \times \{0\} \cup f(\partial I^k) \times I$. Hence $(h_t f): (I^k, \partial I^k) \to (X, X\backslash A)$ is an ε-homotopy with $h_0 f = f$ and $h_1 f(I^k) \subset X\backslash A$; thus A is l.h. negligible in X. □

For $0 \le k \le \infty$ let us say that A is a Z_k*-set* in X if each map $f: I^k \to X$ can be approximated by maps into $X\backslash A$. It is easy to see that A is a Z_∞-set in X iff it is a Z_k-set for all $k \in \mathbb{N}$; closed Z_∞-sets in X will be called *Z-sets*.

PROPOSITION 3.3. *Let* $X \in LC^n$ *be a metric space. Then the following conditions on a closed set* $A \subset X$ *are equivalent:*

(a) $X\backslash A$ *is dense in* X *and is* LC^{n-1} *rel.* X *at each* $x \in A$;

(b) A *is a* Z_n*-set in* X;

(c) A *is locally* n*-negligible in* X.

PROOF. (a) ⇒ (b) follows from theorem 2.8 and (c) ⇒ (a) is trivial. Finally, (b) ⇒ (c) follows by well-known properties of LC^n-spaces (see [21,p.160]). □

In particular, if X has an open basis consisting of homotopy trivial sets then, by proposition 3.3 and theorem 2.3, the closed l.h. negligible sets in X coincide with the Z-sets in X and also with the closed subsets of X which satisfy the following condition (Z) of R.D. ANDERSON [1]: *for each non-empty open homotopy trivial set* U *in* X, $U\backslash A$ *is non-empty open and homotopy trivial.*

4. ENLARGING AN ANR($\mathcal{M}$)-OPEN QUESTIONS AND REMARKS

Let X be a locally contractible metric space and A its l.h. negligible subset. By proposition 3.2, $X \in ANR(\mathcal{M}) \Rightarrow X \setminus A \in ANR(\mathcal{M})$. We do not know whether the converse implication is true.

PROBLEM 4.1. Let X be a metric LC^∞-space, let A be its l.h. negligible subset and assume that $X\setminus A \in ANR(\mathcal{M})$. Is then X an ANR($\mathcal{M}$)? What about the special cases where, in addition,

(a) X is a topological group and A is of type F_σ in X, or

(b) X is compact and $X\setminus A$ is homeomorphic to an open subset of the Hilbert cube I^∞?

Taking products with an appropriate space one can always assume in the general case and in the case (a) that $X\setminus A$ is topologically an open subset of a normed linear space. To show how problem 4.1 is related to some other questions of infinite-dimensional topology let us observe the following two facts.

PROPOSITION 4.2. *Let* X *be a metric space and* A *its* ANR($\mathcal{M}$)-*subset. Then* A *may be enlarged to an* ANR($\mathcal{M}$)- *set* $\tilde{A} \subset X$ *which is of type* G_δ *in* X *and has the property that* $\tilde{A}\setminus A$ *is l.h. negligible in* $\tilde{A}$.

PROOF. By well-known properties of ANR($\mathcal{M}$)'s there is a $\mathcal{U} \in \mathrm{cov}(A\times(0,1])$ and a map $g: |K| \to A$, where K is the nerve of $\mathcal{U}$, such that if $f: A \times (0,1] \to |K|$ is any canonical map then $\rho(gf(x,t),x) < t$ for all $(x,t) \in A \times (0,1]$ (see [21,p.138] or use the proof of theorem 1.1). Let $\mathcal{V}$ be a family of open subsets of $X \times (0,1]$ such that $\mathcal{U} = \{V \cap A\times (0,1] : V \in \mathcal{V}\}$, let L be the nerve of $\mathcal{V}$ and V be the union of all elements of $\mathcal{V}$, and let $f: V \to |L|$ be a canonical map. Identifying K with a subcomplex of L we infer that $C = f^{-1}(|K|)$ is a relatively closed subset of V and therefore the set $B = \{(x,t) \in C : \rho(gf(x,t),x) < t\}$ is of type G_δ in $X \times (0,1]$ and contains $A \times (0,1]$. Since $(0,1]$ is σ-compact, $\tilde{A} = X\setminus p_X(X\times(0,1]\setminus B)$ is a G_δ-subset of X which clearly contains A. By lemma 1.4, $\tilde{A} \in ANR(\mathcal{M})$. The following sublemma yields that $\tilde{A}\setminus A$ is l.h. negligible in $\tilde{A}$.

SUBLEMMA. *Every set* $T \subset X\backslash \mathrm{im}(g)$ *is l.h. negligible in* X.

PROOF. Let $h: I^n \to X$ be given. Identify I^n with $I^n \times \{0\} \subset I^n \times I$. By lemma 1.4 there are $\varepsilon > 0$ and $\bar{h}: I^n \times [0,\varepsilon] \to X$ such that $\bar{h}|I^n = h$; moreover, the formula given in the proof of lemma 1.4 yields $\bar{h}(x,t) \in \mathrm{im}(g)$ for $t \in (0,\varepsilon]$. Therefore there is a homotopy $(u_t): I^n \to X$ such that $u_0 = h$ and $u_t(I^n) \cap T = \emptyset$ for $t > 0$; this concludes the proof. □

Let us notice that the set $\tilde{A}$ of proposition 4.2 is in no way unique; e.g. if $B \supset A$ is any G_δ-subset of $\tilde{A}$ then B also satisfies the assertion of proposition of 4.2, see corollary 2.6 and section 3.

LEMMA 4.3. *If* $f: I^\infty \to Y$ *is a surjection such that all the sets* $f^{-1}(y)$, $y \in Y$, *have trivial shape, then* Y *is a l.h. negligible set in the mapping cylinder* Z_f *of* f. The proof is given in the Appendix.

Now, we shall illustrate problem 4.1 by the following examples.

EXAMPLE 4.4. Let $f: I^\infty \to Y$ be a surjection with pre-images of points being of trivial shape. Letting $(X,A) = (Z_f,Y)$ we infer, by lemma 4.3, that the affirmative answer to problem 4.1(b) would imply that Z_f and hence Y are AR's. (This in turn would imply that cell-like maps preserve shapes of compacta and their property of being an ANR, which is known to be true under certain dimensionality restrictions).

EXAMPLE 4.5. Let E be a separable linear metric space and let E_0 denote the linear span of a dense countable subset of E. By a theorem of HAVER [15], E_0 is an ANR(M)-space and, by lemma 5.3, there is an ANR(M)-set $M \supset E_0$ which is of type G_δ in E. Then $E\backslash E_0$ and hence $A = E\backslash M$ are l.h. negligible in E (see remark 2.9 and corollary 2.6); thus the affirmative answer to problem 4.1(a) would imply that $E \in$ ANR(M).

EXAMPLE 4.6. Let X be a compact PL-manifold, let H denote its homeomorphism group with compact-open topology and let P be the subgroup of H consisting of PL-maps. It was shown by HAVER [15],[17], that $P \in$ ANR(M) and the closure G of P is an open subgroup of H. Let $G_0 \supset P$ be an ANR(M)-extension of P to a G_δ-subset of G; since P is uniformly locally contractible (see [17]) we infer, by remark 2.9 and corollary 2.6, that $G\setminus G_0$ is l.h. negligible in G. Thus $G \times \ell_2$ contains an ℓ_2-manifold (namely $G_0 \times \ell_2$, see [31]) with an l.h. negligible complement. Since H is a union of open cosets of G and since $H \times \ell_2 \cong H$ (GEOGHEGAN [14]), H also contains an ℓ_2-manifold with an l.h. negligible complement. Thus the affirative answer to problem 4.1(a) would imply that H is an ANR(M).

5. ENLARGING A MANIFOLD

In this section we show that if X is a complete ANR(M)-space which contains an ℓ_2-manifold whose complement is a Z-set in X, then X is necessarily an ℓ_2 manifold. We start with

PROPOSITION 5.1. *Let* E *denote the Hilbert cube or a locally convex linear metric space such that* $E \cong E^\infty$ *or* $E \cong \Sigma E = \{(x_i) \in E^\infty : x_i = 0$ for almost all $i\}$ *and let* A *be a Z-set in a metric space* X. *If both* $X \times E$ *and* $X\setminus A$ *are* E-*manifolds then* $X \cong X \times E$ *and (in particular)* X *is an* E-*manifold.*

The proof is divided into 3 steps and involves an idea of CUTLER (see [7] and also [33], where some special cases of proposition 5.1 are established).

(1^o) *If* M *is an* E-*manifold and* K *is a* Z-*set in* M *then there is a homotopy* $(f_t): M \to M$ *such that* $f_0 = \mathrm{id}$, $f_t(M) \subset \mathrm{int}\, f_s(M)$ *if* $0 < s < t \le 1$, $\bigcup_{t>0} f_t(M) = M\setminus K$ *and* $(x,t) \to (f_t(x),t)$ *is a closed embedding of* $M \times I$ *into itself.*

PROOF. Under our assumptions there is a homeomorphism $h: M \xrightarrow{\text{onto}} M \times I$ such that $h(K) \subset M \times \{0\}$ (see [30]). Let ρ be any product metric on $M \times I$; then for each $t \in I$ the formula

$$\alpha_t(x) = \inf\{s \in I : \rho((x,s),h(K)) \ge t\}$$

defines a continuous function on M. We let $f_t = h^{-1} g_t h$, where $g_t(x,s) = (x,s)$ if $s \ge \alpha_t(x)$ and $g_t(x,s) = (x, \frac{1}{2}\alpha_t(x) + \frac{s}{2})$ otherwise.

Given spaces Z and F and a closed set $L \subset Z$ we denote by $(Z\times F)_L$ the space $(Z\setminus L) \times F \cup L$ equipped with the topology generated by open subsets of $(Z\setminus L) \times F$ and be sets of the form $U \cap L \cup (U\setminus L) \times F$, where $U \subset Z$ is open. CF denotes $(I\times F)_{\{0\}}$, the cone over F. □

(2°) *Under the assumptions of proposition 5.1, the spaces* $X \times CE$ *and* $(X\times CE)_A$ *are homeomorphic.*

PROOF. Set $M = X \times E$ and $K = A \times E$ and let $(f_t)\colon M \to M$ be the homotopy from (1°). Define $h\colon X \times CE \to (X\times CE)_A$ by the formula

$$h(x,y) = \begin{cases} (x,y) & \text{if } y = 0, \\ \left(p_X f_t(x,e), \left(p_E f_t(x,e), \dfrac{t}{\beta f_t(x,e)}\right)\right) & \text{if } y = (t,e) \text{ and } t > 0, \end{cases}$$

where $\beta(x,e) = \sup\{s \in I : (x,e) \in f_s(M)\}$. It is a matter of a routine but tedious verification that h is a homeomorphism of $X \times CE$ onto $(X\times CE)_A$. □

PROOF OF PROPOSITION 5.1. It is known that E and CE are homeomorphic (see [18] and [33]), and therefore $X \times E \cong (X\times E)_A$. Let ρ be a metric for X. Since $X\setminus A$ is an E-manifold there is a homeomorphism $g\colon (X\setminus A) \times E \xrightarrow{\text{onto}} X\setminus A$ such that $\rho(g(z),p_X(z)) < \rho(p_X(z),A)$ for all $z \in (X\setminus A) \times E$ (see [28]). Extending g by indendity over A we get a homeomorphism of $(X\times E)_A$ onto X. Thus $X \times E \cong X$. □

Combining proposition 5.1 with results of [32] we get

THEOREM 5.2. *Let* X *be an* ANR(M)*-space, let* A *be a* Z*-set in* X *and assume that* $X\setminus A$ *is a manifold modelled on a space* E. *In any of the following cases* X *also is an* E*-manifold:*

(a) E *is an infinite-dimensional Hilbert space and* X *is complete;*

(b) E *is a locally convex linear metric space with* $E \cong \Sigma E$ *and* X *admits a closed embedding into* E.

For a discussion of certain special cases in which the condition (b) is satisfied see [31,§1].

In the remaining part of this section we apply theorem 5.2 to show that certain function spaces are ℓ_2-manifolds. If X is a space and A is a compactum then C(A,X) denotes the space of maps of A into X (compact-open topology), for $x \in X$ we denote by $\hat{x}$ the constant map with value x, and we let $\hat{X} = \{\hat{x} : x \in X\}$. $C((A,A_0),(X,X_0))$ has the usual meaning. We need two lemmas leading to the fact that if $X \in ANR(M)$ has no isolated points, then one can continuously assigne to each $x \in X$ a non-constant path starting from x.

LEMMA 5.3. *Let* $Y \in ANR(M)$, *let* $A_0 \subset A$ *be compacta and let* $y_0 \in Y$. *If neither* $\{y_0\}$ *nor* A_0 *are open then the singleton* $\{\hat{y}_0\}$ *is a Z-set in* $S =$
$= C((A,A_0),(Y,y_0))$,

PROOF. Since every $f \in S$ factorizes through a map of $(A/A_0,[A])$ into (Y,y_0), we may assume that $A_0 = \{a_0\}$ is a one-point set. Consider A as a (nowhere-dense) subset of ℓ_2 and let $(a_n) \in (A\setminus A_0)^\infty, (z_n) \in (\ell_2\setminus A)^\infty$ and $(y_n) \in (Y\setminus\{y_0\})^\infty$ be a sequences such that $\lim z_n = \lim a_n = a_0$ and $\lim y_n = y_0$. Given $f: A \times I^\infty \to Y$ with $f(\{a_0\} \times I^\infty) \subset \{y_0\}$, extend f to $f_1: (A \cup \{z_n : n \in N\}) \times I^\infty \to Y$ by letting $f_1(\{z_n\} \times I^\infty) = \{y_n\}$, $n \in N$, and extend f_1 to an $\bar{f}: U \times I^\infty \to Y$ where $U \supset A \cup \{z_n : n \in N\}$ is open in ℓ_2. Let (g_n) be a sequence of mappings $g_n: A \to U$ such that $\lim g_n = id$ and, moreover, $g_n(a_0) = a_0$ and $g_n(a_n) = z_n$ for all sufficiently big n's. Then the maps $f_n: A \times I^\infty \to Y$ defined by

$$f_n(a,q) = f(g_n(a),q), \quad (a,q) \in A \times I^\infty, n \in N,$$

converge to f and have the property that, for each $q \in I^\infty$, the map $a \to f_n(a,q)$ belongs to $S\setminus\{\hat{y}_0\}$. Since $f: A \times I^\infty \to Y$ was induced by an arbitrary map of I^∞ into S, the result follows. □

LEMMA 5.4. *Let* Y *be an* ANR(M)*-space without isolated points and let* $\varepsilon > 0$. *Then there is a* $v: Y \to C(I,Y)\setminus\hat{Y}$ *such that* $v(y)(0) = y$ *and* $\hat{\rho}(v(y),\hat{y}) < \varepsilon$ *for all* $y \in Y$ ($\hat{\rho}$ denotes here the sup-metric induced by ρ).

PROOF. C(I,Y) is an ANR(M)-space and therefore, by theorem 2.4, proposition 3.3 and elementary properties of ANR(M)'s, it suffices to show that $C(I,Y)\setminus\hat{Y}$ is LC^∞ rel. C(I,Y) at each point $\hat{y} \in \hat{Y}$ (we omit the verification that $C(I,Y)\setminus\hat{Y}$ is dense in C(I,Y)).

To this end let us fix $k \in N$, $\hat{y}_0 \in \hat{Y}$ and $\varepsilon_0 > 0$; we shall find a $\delta > 0$ such that, under notation $S = C(I,Y)$ and $J = [-1,1]$, we have

$$(*)\begin{cases}\text{Each } f\colon \partial J^k \to S\setminus\hat{Y} \text{ with } \sup\{\hat{\rho}(f(x),\hat{y}_0) : x \in \partial J^k\} < \delta \\ \text{extends to an } \bar{f}\colon J^k \to S\setminus\hat{Y} \text{ with } \sup\{\hat{\rho}(\bar{f}(x),\hat{y}_0) : x \in J^k\} < \varepsilon_0.\end{cases}$$

First observe that, by lemma 5.3 and proposition 3.2, there is a $\delta_0 > 0$ such that each $g\colon \partial J^k \to C(([0,2],2),(Y,y_0))\setminus\{\hat{y}_0\}$ with $\hat{\rho}(g(x),\hat{y}_0) < 2\delta_0$ for $x \in \partial J^k$ admits an extension $g\colon J^k \to C(([0,2],2),(Y,y_0))\setminus\{\hat{y}_0\}$ with $\rho(g(x),\hat{y}_0) < \varepsilon_0$ for $x \in J^k$. Since $Y \in \mathrm{ANR}(M)$, there exists further a $\delta > 0$ such that the δ-ball of Y centered at y_0 can be deformed to y_0 inside the δ_0-ball centered at y_0. We shall show that δ satisfies $(*)$. Indeed, if f is as in $(*)$, then there exists a $w\colon J^k \to Y$ with $w(x) = f(x)(1)$ for $x \in \partial J^k$, $w(0) = y_0$, and $\rho(w(x),y_0) < \delta_0$ for $x \in J^k$. Letting

$$g(x)(t) = \begin{cases} f(x)(t) & \text{for } t \in [0,1], \\ w((2-t)x) & \text{for } t \in [1,2], \end{cases}$$

we get a $g\colon \partial J^k \to C(([0,2],2),(Y,y_0))$ with $\hat{\rho}(g(x)\hat{y}_0) < 2\delta_0$ for $x \in \partial J^k$. Since $\hat{y}_0 \notin \mathrm{im}(g)$, g admits an extension $\bar{g}\colon J^k \to C(([0,2],2),(Y,y_0))\setminus\{\hat{y}_0\}$ with $\hat{\rho}(\bar{g}(x),\hat{y}_0) < \varepsilon_0$ for all $x \in J^k$. If we let $h_r(t) = (-2r+3)t$, $t \in I$, then

$$\bar{f}(rx) = \begin{cases} g(x) \circ h_r & \text{if } x \in \partial J^k, r \in [1/2,1], \\ g(2r\cdot x) \circ h_{\frac{1}{2}} & \text{if } x \in J^k, r \in [0.1/2], \end{cases}$$

defines the extension required in $(*)$. □

<u>THEOREM 5.5</u>. *Let* X *and* $X_1,\dots,X_n \subset X$ *be separable complete* ANR(M)'s, *let* A *be a compactum and* $A_1,\dots,A_n$ *its disjoint closed subsets, and let* U *be a cone-patch* *) *for* A. *If either* $U \cap (A_1\cup\dots\cup A_n) = \emptyset$ *and* X *has no isolated points or* $U \subset A_1$ *and* X_1 *has no isolated points, then the space*

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*) The terminology is that of GEOGHEGAN [14].

$S = \{f \in C(A,X) : f(A_i) \subset X_i \text{ for } i=1,2,\ldots,n\}$ *is an* ℓ_2*-manifold.*

PROOF. Let $K = \{f \in S : f \text{ is constant on } U\}$. It is known that $S\setminus K$ is an ℓ_2-manifold and S is a complete separable ANR(M)-space (see [31,§4]). Therefore it remains to show that K is a Z-set in S.

To this end fix $\varepsilon > 0$ and $f: I^\infty \times A \to X$ such that $f_q = f(q,\cdot) \in S$ for all $q \in I^\infty$. By the definition of cone-patches there exists an $a_0 \in U$ and a homotopy $(u_t): A \to A \times \{0\} \cup \{a_0\} \times I$ such that $u_t(a) = (a,0)$ if $a \notin U$ or $t = 0$ and $u_t(A) = A \times \{0\} \cup \{a_0\} \times [0,t]$ for all $t \in I$. Define $\tilde{f}: I^\infty \times (A\times\{0\}\cup\{a_0\}\times I) \to X$ by

$$\tilde{f}(q,z) = \begin{cases} f(q,z) & \text{if } q \in I^\infty,\ z \quad A = A \times \{0\}, \\ v(f(q,a_0))(t) & \text{if } q \in I^\infty,\ z = (a_0,t) \in \{a_0\} \times (0,1], \end{cases}$$

where v satisfies lemma 5.4 with $Y = X$ in case $U \cap (A_1 \cup \ldots \cup A_n) = \emptyset$ and with $Y = X_1$ in case $U \subset A$. Choose $\delta > 0$ such that $\hat{\rho}(\tilde{f}(u_\delta \times id), f) < \varepsilon$ and define $g: I^\infty \times A \to X$ by

$$g(q,a) = \begin{cases} \tilde{f}(q,u_\delta(a)) & \text{if } u_\delta(a) \in A \times \{0\}, \\ \tilde{f}(q,(a_0,t/\delta)) & \text{if } u_\delta(a) = (a_0,t) \in \{a_0\} \times I. \end{cases}$$

One easily verifies that $g_q = g(q,\cdot) \in S\setminus K$ and $\hat{\rho}(g_q,f_q) < 2\varepsilon$ for all $q \in I^\infty$. This shows that K is a Z_∞-set in Y. □

COROLLARY 5.6. *Let* X *and* $X_1,\ldots,X_n \subset X$ *be complete separable* ANR(M)'s, *where* X *has no isolated points. If* A *is a connected compact finite-dimensional manifold (possibly with boundary), then for any closed mutally disjoint proper subsets* $A_1,\ldots,A_n$ *of* A *the space* $\{f \in C(A,X) : f(A_i) \subset X_i$ for $i=1,2,\ldots,n\}$ *forms an* ℓ_2*-manifold. In particular the space of paths from* X_1 to X_2 *and the space of closed curves starting from* X_1 *are* ℓ_2*-manifolds.*

6. APPENDIX. LOCALLY HOMOTOPY NEGLIGIBLE SETS AND UV^{∞}-MAPS

We shall show here how the properties of l.h. negligible sets are related to the results of ARMENTROUT-PRICE, KOZLOWSKI and LACHER on cell-like mappings of metric spaces.

All spaces are assumed to be metrizable. If $f: X \to Y$ is a map then by the mapping cylinder of f we mean the space $Z_f = X \times [0,1) \cup Y \times \{1\}$, equipped with the topology generated by open subsets of $X \times [0,1)$ and by sets $f^{-1}(U) \times (t,1) \cup U \times \{1\}$, where $t > 0$ and $U \subset Y$ is open. Note that Z_f is metrizable: if we consider X and Y as bounded subsets of normed spaces E and F respectively, then $Z_f \cong \{0\} \times \{1\} \times Y \cup \cup \{(x-tx, t, t\cdot f(x)) : t \in I, x \in X\} \subset E \times I \times F$. We identify X with $X \times \{0\}$, Y with $Y \times \{1\}$ and we denote by $p: Z_f \to Y$ and $q: Z_f \setminus Y \to X$ the collapse and projection respectively.

A map $f: X \to Y$ will be said to be UV^n if, given $y \in Y$, $k < n+1$ and a neighbourhood U of y, there is a neighbourhood $V \subset U$ of y such that each $g: \partial I^k \to f^{-1}(V)$ extends to a $g: I^k \to f^{-1}(U)$. If $A \subset X$ is a compactum and the projection $X \to X\setminus A$ is a UV^n-map, then we say that A is a UV^n-subset of X.

REMARK 6.1. f is a UV^n-mapping iff $Z_f \setminus Y$ is LC^n rel. Z_f at each point of Y. If all the $f^{-1}(y)$'s are compact and f is a surjection then f is a UV^n-map iff all the $f^{-1}(y)$'s, $y \in Y$, are UV^n-subsets of X.

It is known that compacta of trivial shape are UV^{∞}-subsets of ANR($\mathcal{M}$)'s in which they lie (see [5]).

PROPOSITION 6.2. (Compare [27],[22],[4]). *If* $f: X \to Y$ *is a* UV^n*-map and* $f(X)$ *is dense in* Y *then* f *induces an isomorphism of the* n*-th homotopy group.*

PROOF. Apply theorem 2.8 and the fact that f induces an isomorphism of the n-th homotopy group iff the inclusion $Z_f \setminus Y \to Z_f$ does. □

PROPOSITION 6.3. *Let* $f: X \to Y$ *be a* UV^{∞}*-map with dense image and let* $M \in$ ANR($\mathcal{M}$). *Then, given* $u: M \to Y$ *and* $\alpha: M \times (0,1] \to (0,\infty)$, *there is a* $g: M \times (0,1] \to X$ *such that* $\rho(fg_t(x), u(x)) < \alpha(x,t)$ *for* $(x,t) \in M \times (0,1]$.

If, in addition, $K \subset X$ *is a closed set,* U *is its neighbourhood and* $v\colon U \to X$ *is any lifting of* $u|U$, *then* g *may be constructed in such a way that* $g_t|K = v|K$ *for all* t.

PROOF. Put on Z_f a metric d in which the collapse $p\colon (Z_f,d) \to (Y,\rho)$ is a contraction and let $\lambda\colon M \to [0,1]$ satisfy $\lambda|K = 1$ and $M\setminus U \subset \operatorname{int} \lambda^{-1}(0)$. Define $w\colon M \to Z_f$ by

$$w(x) = \begin{cases} (v(x),\lambda(x)) \in X \times [0,1) & \text{if } \lambda(x) > 0, \\ u(x) & \text{if } \lambda(x) = 0. \end{cases}$$

Since, by theorem 2.8, Y is l.h. negligible in Z_f, there exists an λ-homotopy $(h_t)\colon M \to Z_f$ such that $h_t(M) \subset Z_f\setminus Y$ and $h_t|K = w$ for all $t > 0$. We let $g_t = qh_t$. □

PROPOSITION 6.4. *Let* $f\colon X \to Y$ *be an* UV^∞*-map of* ANR(M)'s *and assume that* $f(X)$ *is dense in* Y. *Then, given* $\alpha\colon Y \times (0,1] \to (0,\infty)$, *there exist* $g\colon Y \times (0,1] \to X$ *and a homotopy* $(h_t)\colon X \to X$ *such that* $h_0 = \mathrm{id}$, $h_1 = g_1 f$ *and* $\rho(fg_t(y),y) < \alpha(y,t)$ *and* $\rho(fg_t(x),f(x)) < \alpha(f(x),t)$ *for all* $t \in (0,1]$, $x \in X$, $y \in Y$.

PROOF. Let λ be any increasing homeomorphism of $[-1,2]$ onto $[0,1]$. By proposition 6.2 there is a $g\colon Y \times (0,1] \to X$ such that, for all $(y,t) \in Y \times (0,1]$,

$$\rho(fg_t(y),y) < \tfrac{1}{2} \min(\alpha_t(y),t,\alpha_{\lambda(t)}(y),\inf\{\alpha_s(y):s \in \lambda([1,2])\}).$$

Let $M = X \times [-1,2]$, $K = X \times \{-1,2\}$, $U = X \times ([-1,0) \cup (1,2])$, and define $u\colon M \to Y$ by $u_t = f$ for $t \in [-1,0]$, $u_t = fg_t f$ for $t \in (0,1]$, and $u_t = fg_1 f$ for $t \in [1,2]$.

Using proposition 6.2 again, construct $\tilde{h}\colon M \to X$ with $\tilde{h}_{-1} = \mathrm{id}$, $\tilde{h}_2 = g_1 f$ and $\rho(fh_t(x),u_t(x)) < \tfrac{1}{2}\, \alpha_{\lambda(t)}(f(x))$ for $(t,x) \in M = X \times [-1,2]$. Finally, let $h_t = h_{\lambda-1(t)}$. □

REMARK 6.5. Let $f: X \to Y$ be a UV^n-map with dense image and assume that X is an LC^n-space and $\dim(Y) \le n$. It follows easily from remark 6.1 and theorem 2.8 that Y is LC^n and therefore $Y \in ANR(M)$ by [21,p.168].

REMARK 6.6. Let $f: X \to Y$ be an UV^{n-1}-map with dense image and assume that X and Y are ANR(M)'s and $\max(\dim(Y), \dim(X)+1) \le n < \infty$. Then, $\dim(Z_f) \le n$ and Z_f is locally contractible, and therefore $Z_f \in ANR(M)$ (see [21,p.168]). Hence, by proposition 3.2, Y is l.h. negligible in Z_f and f is actually a UV^∞-map; thus proposition 6.4 applies.

We also observe that if X and Y are locally compact spaces and f is a proper map then the homotopies $id \cup (fg_t)_{t>0}$ and (h_t) of proposition 6.4 are proper if α is taken sufficiently small (slightly weaker versions of remarks 6.5 and 6.6 form the theorems of LACHER [23]).

COROLLARY 6.7. *Let* $f: X \to Y$ *be a surjection such that all the* $f^{-1}(y)$'s, $y \in Y$, *are compact* UV^n-*subsets of* X. *If* X *is an* n-*dimensional* ANR(M)- *space and* Y *is finite-dimensional then* $Y \in ANR(M)$ *and* f *is a* UV^∞-*map*.

PROOF. Fix $y_0 \in Y$ and consider the quotient map $\pi: X \to X\backslash f^{-1}(y_0) = S_{y_0}$. By remark 6.5 we have $S_{y_0} \in ANR(M)$ and therefore, by remark 6.6, π is a UV^∞-map. Thus all the $f^{-1}(y)$'s , $y \in Y$, are UV^∞-subsets of X and the assertion follows from remarks 6.1 and 6.5. □

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